# Lista de soluções 8

**Exercício 1 (Laço while)**

(a) A equação particular para `while ¬(x=0) do skip` é

$$\begin{aligned} C[\![\texttt{while ¬(x=0) do skip}]\!] &= \mathrm{cond}(B[\![\texttt{¬(x=0)}]\!], [\![\texttt{while ¬(x=0) do skip}]\!] \circ C[\![\texttt{skip}]\!], \mathrm{id}) \\ &= \mathrm{cond}(B[\![\texttt{¬(x=0)}]\!], [\![\texttt{while ¬(x=0) do skip}]\!], \mathrm{id}) \\ &= \mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, [\![\texttt{while ¬(x=0) do skip}]\!], \mathrm{id}) \end{aligned}$$

(b) No caso de $g_1$ temos que verificar a equação

$$g1 = \mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g_1, \mathrm{id}).$$

Obtemos

$$\mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g_1, \mathrm{id})\sigma = \begin{cases} g_1\sigma & \text{se } \sigma(x) \neq 0 \\ \sigma & \text{se } \sigma(x) = 0 \end{cases}$$

e logo $\mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g_1, \mathrm{id}) = g_1$. No caso de $g_2$ temos que verificar a equação

$$g2 = \mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g_2, \mathrm{id}).$$

Obtemos

$$\mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g_1, \mathrm{id})\sigma = \begin{cases} \sigma' & \text{se } \sigma(x) \neq 0 \\ \sigma & \text{se } \sigma(x) = 0 \end{cases}$$

e logo $\mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g_2, \mathrm{id}) = g_2$.

(c) Considerando o laço, o significado deve ser $g_1$. A ordenação das funções $g_1 \sqsubseteq g_2$ corresponde com isso.

(d) No caso de $g$ temos que verificar a equação

$$g = \mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g, \mathrm{id}).$$

Obtemos

$$\mathrm{cond}(\lambda\sigma.\sigma(x) \neq 0, g, \mathrm{id})\sigma = \begin{cases} \perp & \text{se } \sigma(x) \neq 0 \\ \sigma & \text{se } \sigma(x) = 0 \end{cases}$$

e logo para estados $\sigma$ tal que $\sigma(x) = 0$, os dois lados são diferentes.

**Exercício 2 (Laço while)**

Para `while x>0 do x:=x-1` obtemos a equação

$$C[\![\texttt{while x>0 do x:=x-1}]\!] = \mathrm{fix}\, F$$

com

$$\begin{aligned} F(\tau) &= \mathrm{cond}(B[\![x > 0]\!], \tau \circ C[\![x := x - 1]\!], \mathrm{id}) \\ &= \mathrm{cond}(\lambda\sigma.\sigma(x) > 0, \tau \circ \lambda\sigma.\sigma[x \mapsto \sigma(x) - 1], \mathrm{id}) \end{aligned}$$

(Observação: Na folha dos exercícios o exemplo foi erradamente `while x>0 do skip`).
Vamos provar com indução natural a característica

$$P(n) : F^n(\bot) = \begin{cases} \lambda\sigma \in \Sigma.\sigma & \text{se } \sigma(x) \not> 0 \\ \lambda\sigma \in \Sigma.\sigma[x \mapsto 0] & \text{se } 1 \le \sigma(x) < n \\ \lambda\sigma \in \Sigma.\bot_{[\Sigma_\bot \to \Sigma_\bot]} & \text{se } \sigma(x) \ge n \end{cases}$$

Base: No caso $n = 1$

$$\begin{aligned} F^n(\bot) = F(\bot) &= \mathrm{cond}(\lambda\sigma.\sigma(x) > 0, \bot \circ \lambda\sigma.\sigma[x \mapsto \sigma(x) - 1], \mathrm{id}) \\ &= \mathrm{cond}(\lambda\sigma.\sigma(x) > 0, \bot, \mathrm{id}) \\ &= \begin{cases} \lambda\sigma \in \Sigma.\sigma & \text{se } \sigma(x) \not> 0 \\ \lambda\sigma \in \Sigma.\bot_{[\Sigma_\bot \to \Sigma_\bot]} & \text{se } \sigma(x) \ge 0 \end{cases} \end{aligned}$$

que é a mesmo função de acima. Passo: Suponhe o caso $P(n)$ acima. Então $F^{n+1}(\bot)\sigma = F(F^n(\bot))\sigma$, e temos os casos

(i) $\sigma(x) \not> 0$: $F(F^n(\bot))\sigma = F^n(\bot)\sigma = \sigma$

(ii) $\sigma(x) > 0$: $F(F^n(\bot))\sigma = F^n(\bot) \circ (\lambda\sigma \in \Sigma.\sigma[x \mapsto \sigma(x) - 1])\sigma = F^n(\bot)\sigma[x \mapsto \sigma(x) - 1]$. Temos os seguintes subcasos:

  (a) $\sigma(x) = 1$. Logo $F^n(\bot)\sigma[x \mapsto \sigma(x) - 1] = F^n(\bot)\sigma[x \mapsto 0] = \sigma[x \mapsto 0]$.

  (b) $1 < \sigma(x) \le n$. Logo $F^n(\bot)\sigma[x \mapsto \sigma(x) - 1] = \sigma[x \mapsto 0]$

  (c) $n < \sigma(x)$. Logo $F^n(\bot)\sigma[x \mapsto \sigma(x) - 1] = \bot$.

  Com todos subcasos juntos, obtemos $P(n + 1)$.

**Exercício 3 (Laço repeat)**
Podemos usar a equivalência

$$\texttt{repeat}\ \mathbf{c}\ \texttt{until}\ \mathbf{b} \equiv \mathbf{c}\texttt{;if}\ \mathbf{b}\ \texttt{then skip else repeat}\ \mathbf{c}\ \texttt{until}\ \mathbf{b}.$$

Com isso obtemos

$$\begin{aligned} C[\![\texttt{repeat}\ \mathbf{c}\ \texttt{until}\ \mathbf{b}]\!] &= C[\![\mathbf{c}\texttt{;if}\ \mathbf{b}\ \texttt{then skip else repeat}\ \mathbf{c}\ \texttt{until}\ \mathbf{b}]\!] \\ &= C[\![\texttt{if}\ \mathbf{b}\ \texttt{then skip else repeat}\ \mathbf{c}\ \texttt{until}\ \mathbf{b}]\!] \circ C[\![\mathbf{c}]\!] \\ &= \mathrm{cond}(B[\![\mathbf{b}]\!], \mathrm{id}, \texttt{repeat}\ \mathbf{c}\ \texttt{until}\ \mathbf{b}) \circ C[\![\mathbf{c}]\!] \end{aligned}$$

ou em notação com ponto fixo

$$C[\![\texttt{repeat}\ \mathbf{c}\ \texttt{until}\ \mathbf{b}]\!] = \mathrm{fix}\, F$$

com

$$F(\tau) = \mathrm{cond}(B[\![b]\!], \mathrm{id}, \tau) \circ C[\![c]\!].$$